

Fast plurality consensus in regular expanders *

Colin Cooper[†] Tomasz Radzik[‡] Nicolás Rivera[§] Takeharu Shiraga[¶]

April 14, 2017

Abstract

The problem of reaching consensus in a graph by means of local interactions is an abstraction of such behavior in human society as well as some distributed processes in computer networks. In a *voting process* on a graph vertices revise their opinions in a distributed way based on the opinions of nearby vertices. The classic example is synchronous pull voting where at each step, each vertex adopts the opinion of a random neighbour. This type of pull voting suffers from two main drawbacks. Even if there are only two opposing opinions, the time taken for a single opinion to emerge can be slow, and the final opinion is not necessarily the initial majority. Things can often be improved by using a variant of synchronous pull voting in which each vertex considers the opinions of two neighbours. For many classes of n -vertex regular expanders, consensus is now reached in $O(\log n)$ expected steps [8], as opposed to $\Theta(n)$ expected steps [7] when only one neighbour is contacted. Moreover, this protocol allows the initial majority opinion to win with high probability.

In the case where there are initially three or more opinions, not so much is known about the performance of voting using two or more samples. A problem arises when there is no clear majority. Thus one class of opinions may be largest, but its total size is less than that of two other opinions put together. When there are three or more opinions, the term *plurality* is often used to distinguish this case from that of an overall majority.

In the case where the underlying network is the complete graph K_n , Becchetti et. al [3, 4] analysed the general case of $k \geq 3$ opinions using a *three-sample voting process* and proved the following result. Let A_1 be the initial size of the largest opinion. Then if the difference between the initial sizes of the largest and second largest opinions is at least $Cn\sqrt{(\log n)/A_1}$, for some suitable constant C , the largest opinion wins in $O((n \log n)/A_1)$ steps with high probability.

In this paper we show that similar performance can be achieved on d -regular expanders using *two-sample voting*. Namely, if the difference between the initial sizes of the largest and second largest opinions is at least $Cn \max\{\sqrt{(\log n)/A_1}, \lambda\}$, for some suitable constant C , then the largest opinion wins in $O((n \log n)/A_1)$ steps with high probability. Here λ is the absolute second eigenvalue of transition matrix $P = \text{Adj}(G)/d$ of a simple random walk on the graph G . For almost all d -regular graphs, we have $\lambda = c/\sqrt{d}$ for some constant $c > 0$ [12]. Thus as d increases we can separate an opinion whose plurality is $o(n)$, whereas a plurality of $\Theta(n)$ appears to be needed for d constant. Finally for d constant we show how this $\Theta(n)$ barrier can be reduced by sampling using short random walks.

*This work was supported in part by EPSRC grant EP/M005038/1, “Randomized algorithms for computer networks”. Nicolás Rivera was supported by funding from Becas CHILE. Takeharu Shiraga was supported by JSPS KAKENHI Grant Number 15J03840. Work carried out while Takeharu Shiraga was visiting King’s College London with the support of the ELC project (Grant-in-Aid for Scientific Research on Innovative Areas MEXT Japan).

[†]Department of Informatics, King’s College London, UK. colin.cooper@kcl.ac.uk

[‡]Department of Informatics, King’s College London, UK. tomasz.radzik@kcl.ac.uk

[§]Department of Informatics, King’s College London, UK. nicolas.rivera@kcl.ac.uk

[¶]Theoretical Computer Science Group, Department of Informatics, Kyushu University, Fukuoka, Japan. shiraga@tcslab.csce.kyushu-u.ac.jp

1 Introduction

The problem of reaching consensus in a graph by means of local interactions is an abstraction of such behavior in human society as well as some processes in computer networks. In a *voting process* on a graph, vertices revise their opinions in a systematic and distributed way based on opinions of other vertices, typically using a sample of their local neighbours. The aim is that eventually a single opinion will emerge, and that this opinion will reflect the relative importance of the original mix of opinions in some way.

Voting processes are a natural approach to achieving consensus, and as a consequence they have been widely studied. Distributed voting finds application in various fields of computing including consensus and leader election in large networks [5, 14], serialisation of read/write in replicated data-bases [13], and analysis of social behavior [11]. In general, a voting process should be conceptually simple, fast, fault-tolerant and straightforward to implement [14, 15].

In outline, a voting process can be described as follows. Each vertex of a connected graph has one of several possible opinions. In each time-step, each vertex queries the opinion one or more of its neighbours using the same protocol, and decides whether to modify or to keep its current opinion. When all vertices have a common (and thus final) opinion, we say a consensus has been reached. For a given voting process, the main questions of interest are the probability that a particular opinion wins and the expected time to reach consensus. The most well known model is synchronous pull voting. In this model, at each step each vertex changes its opinion to that of a random neighbour.

In the classical *voter model* each vertex initially has a distinct opinion, but in general we can assume the vertices are restricted to hold one of k different opinions. The simplest case, *two party voting*, is when there are initially two opinions ($k = 2$). If there are at least three opinions ($k \geq 3$) the problem is often referred to as *plurality consensus*. Not so much is known about improving the performance of voting by using two or more samples in the case where there are initially three or more opinions.

If some opinion has an absolute majority, we can group the other opinions together into a single minority class, and use the above two-sample protocol. A problem arises when there is no clear majority. Thus one class of opinions may be largest, but its total size is less than that of two other opinions put together. When there are three or more opinions the term *plurality* is often used to distinguish this case from the overall majority one.

For the problem to be one of plurality consensus, we assume that the initial configuration is such that one opinion is dominant, but there is no overall majority. We might expect that the dominant opinion eventually becomes the final opinion of all vertices. This, however, strongly depends on the voting process. If simple pull voting is used, then given the graph is connected (and aperiodic) the probability that a particular opinion wins is proportional to the initial degree of the opinion in the graph (see [14]). More precisely, if A is the set of vertices initially holding a given opinion, then the probability A wins in the voting process is

$$\Pr(A \text{ wins}) = \sum_{v \in A} \frac{d(v)}{2m} = \frac{d(A)}{2m}, \quad (1)$$

where $d(v)$ is the degree of vertex v and m is the number of edges in the graph. Surprisingly, the probability here depends only on the voting process and not on the initial arrangement of opinions on the graph (any set of vertices of the same total degree would do).

We assume henceforth that the graphs we consider are connected and that the graph is not bipartite,

so that a consensus is possible. For an n -vertex graph, let $\mathbf{ET} = \mathbf{ET}(n)$ be the expected value of the time to consensus T . Much of the early work was on analysing \mathbf{ET} for classical pull voting in an asynchronous model in a continuous time setting. Here the vertices have independent exponentially distributed waiting times (Poisson clocks); see e.g. Cox [10] and Aldous [1]. In the synchronous model the expected time to consensus can be bounded by $\mathbf{ET} = O(H_{\max} \log n)$, where $H_{\max} = O(n^3)$ is the maximum hitting time of any vertex by a random walk; see Aldous and Fill [2]. For regular expanders these results can be improved to $\mathbf{ET} = \Theta(n)$, see [7].

Because the classical pull voting tends to be slow ($\mathbf{ET} = \Theta(n)$ for regular expanders) and may be viewed as undemocratic, there has been considerable interest in modifying this simple voting process to avoid these two problems. Instead of taking the opinion of only one neighbour, the next simplest approach to sample the opinions of a larger number of neighbours (say two or three), compare them in some way, and hope that the so-called ‘power of two choices’ improves the performance of voting. The consequences of this approach are as follows. Firstly, the number of neighbours queried affects the consensus time and the voting outcome. Secondly, the relative size of the opinions affects the ability of the process to ensure that the largest initial opinion wins. Not surprisingly, analysing this relation becomes harder when we move from two party voting to plurality consensus ($k \geq 3$). The additional challenge is that the well established techniques used in analysis of the classical pull voting (for example, the correspondence with multiple coalescing random walks [1, 7]) do not have ready extensions or generalisations to multi-sample voting.

In this setting we study the following protocols for two-sample and three-sample voting. In the two-sample voting model, at each step, each vertex v chooses two random neighbours with replacement, and if the selected vertices have the same the opinion, then v adopts it; otherwise v keeps its current opinion. In the three-sample voting model, each vertex v chooses three random neighbours with replacement, and v adopts the majority opinion among them. If there is no majority, v picks the opinion of the first sampled neighbour. Other rules are equally possible here, e.g. v keeps its opinion. The rule we choose is the one used by Becchetti et. al. [3, 4], and we adopt it for consistency.

Two-sample voting was studied in [8] for the case where there are initially two opinions ($k = 2$). They proved that in d -regular expanders the initial majority wins with high probability (w.h.p.)¹ provided the initial difference between the sizes of the two opinions is sufficiently large, and that voting is completed in $\mathcal{O}(\log n)$ steps. This is tight since the diameter of a d -regular graph is $\Omega(\log n)$ for constant d . In [9] the authors extend the above result to general expander graph, extending the analysis to non-regular graph.

As hinted at above the analysis for plurality consensus ($k \geq 3$) tends to be trickier than for two party voting. This is especially true as k increases, or if two minorities together are much larger than the majority opinion. Plurality consensus using the three-sample voting protocol given above was studied by Becchetti et. al. [3, 4]. They proved that for the complete graph K_n , if the difference between the initial sizes A_1 and A_2 of the largest and second largest opinions is at least $A_1 - A_2 = 24n\sqrt{2(\log n)/A_1}$, then the largest opinion wins in $O((n \log n)/A_1)$ steps w.h.p. They also showed that this result is tight for some ranges of the parameters.

1.1 Our contributions

In this paper we extend the results of [3, 4] from the complete graph to d -regular expanders preserving the same asymptotic convergence time. To do this, we generalize the results of [9] from two-party

¹“With high probability” (w.h.p.) means in this paper probability at least $1 - n^{-\alpha}$, for a constant $\alpha > 0$.

voting to k -party voting. We also give a natural coupling of the three-sample process of [3, 4] with the two-sample process of [9], which allows us to apply our analysis of the two-sample process directly to the three-sample process.

We proceed to state our main result. Let G be a connected regular n -vertex graph and let λ be the second largest absolute eigenvalue of the transition matrix $P = P(G)$ of a random walk on G . Let A_1 be the set of vertices with the largest initial opinion and A_2 the set with the second largest opinion. If no confusion arises, we also use A to stand for the size of set A .

Theorem 1 *Let G be a regular n -vertex graph and let the initial sizes of the opinions be A_1, A_2, \dots, A_k in non-increasing order. Assume that $A_1 - A_2 \geq Cn \max\{\sqrt{(\log n)/A_1}, \lambda\}$, where λ is the absolute second eigenvalue of $P(G)$ and $C > 0$ is a suitably large constant.*

With probability at least $1 - 1/n$, after at most $O((n/A_1) \log(A_1/(A_1 - A_2)) + \log n)$ rounds, the two-sample voting completes and the final opinion is the largest initial opinion.

We note the following w.h.p. property of the second eigenvalue λ for random d -regular graphs for $d = o(n^{1/2})$. For d constant it is a result of Friedman [12] that $\lambda \leq \gamma/\sqrt{d}$, where $\gamma = 2 + \epsilon$ for some small $\epsilon > 0$. For d growing with n , the following estimate of λ is given in [6]. Provided $d = o(n^{1/2})$ there exists constant $\gamma > 0$ such that w.h.p. $\lambda \leq \gamma/\sqrt{d}$. In either case the size separation condition in Theorem 1 is $A_1 - A_2 \geq C'n/\sqrt{d}$.

Theorem 1 can be applied to a number of specific scenarios. Consider, for example, the case where all k opinions are fairly evenly represented, but with one opinion slightly larger than the average n/k . More specifically, assume that $A_1 \geq (n/k)(1 + \epsilon)$, for some $0 < \epsilon \leq 1$, and that $A_2 \leq A_1/(1 + \epsilon)$. Theorem 1 implies the following corollary for this case.

Corollary 1 *For $k \leq ((1/C)^2 n / \log n)^{1/3}$ opinions, if $A_1 \geq (n/k)(1 + \epsilon)$, $A_2 \leq A_1/(1 + \epsilon)$, and $\lambda \leq \epsilon/(Ck)$, where $C > 0$ is the constant from Theorem 1 and $\epsilon^{2/3} = k/((1/C^2)n/\log n)^{1/3} \leq 1$.*

With probability at least $1 - 1/n$, after at most $O(k \log n)$ rounds the two-sample voting completes and the final opinion is the largest initial opinion.

In Section 4 we show that the statements of Theorem 1 and Corollary 1 also hold for the three-sample voting protocol used by Becchetti et. al. [3, 4]. We note that the bound on the running time in Theorem 1 is $O(\log n)$, if A_1 is $\Omega(n/\log n)$, provided that $A_1 - A_2$ is also $\Omega(n/\log n)$ and λ is appropriately small. This improves on the results of [3, 4] which require $A_1 = \Theta(n)$ for a running time of $O(\log n)$.

In the ℓ -extended two-sample voting model, (as introduced in [9]) each vertex makes two independent random walks of length ℓ and carries out two-sample voting using the opinions on the terminal vertices of these walks. By sampling using random walks of length ℓ , we replace the transition matrix P used in the proof of Theorem 1 by its ℓ -th power P^ℓ . If the graph is regular, then the only effect on the proofs is to replace all eigenvalues by their ℓ -th power. This reduces the absolute second eigenvalue from λ to λ^ℓ . By increasing ℓ we can include in our analysis those graphs which do not satisfy the conditions of Theorem 1 on the relation between $A_1 - A_2$ and λ .

Theorem 2 *Let ℓ be a positive integer, let G be a regular n -vertex graph and let the initial sizes of the opinions be A_1, A_2, \dots, A_k in non-increasing order. Assume that $A_1 - A_2 \geq Cn \max\{\sqrt{(\log n)/A_1}, \lambda^\ell\}$, where $C > 0$ is the constant from Theorem 1. Then ℓ -extended two-sample voting completes after at*

most $O((n/A_1)\log(A_1/(A_1 - A_2)) + \log n)$ rounds, with probability at least $1 - 1/n$, and the final opinion is the largest initial opinion.

Once again the same results apply to ℓ -extended three-sample voting.

2 Preliminary Markov chain results

In this section we set up some Markov chain foundations and preliminary results which we need for our proof of Theorem 1. Consider a connected and non-bipartite graph $G = (V, E)$ with n vertices and m edges. Let P be the transition matrix of a simple random walk on G . A random walk on a connected and non-bipartite graph defines a reversible Markov chain with stationary distribution $\pi(x) = d(x)/(2m)$, where $d(x)$ denotes the degree of vertex x . The reversibility of P means that $\pi(x)P(x, y) = \pi(y)P(y, x)$, for all vertices x, y .

Let $1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n > -1$ be the eigenvalues of P and define $\lambda = \lambda(P)$ by $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$. We also consider the matrix $P^2 = P \times P$ (standard matrix product), which is the transition matrix of the two-step random walk, is also reversible and has the same stationary distribution and eigenvectors as P . Moreover, the eigenvalues of P^2 are the squares of the eigenvalues of P . In particular, $\lambda(P^2) = (\lambda(P))^2$. Given $A, B \subseteq V$ and $x \in V$, we define $P(x, A) = \sum_{y \in A} P(x, y)$ and the *flow function* $Q(A, B)$ from A to B as

$$Q(A, B) = \sum_{x \in A} \pi(x)P(x, B). \quad (2)$$

The value of $Q(A, B)$ is the probability that one step of the random walk taken from the stationary distribution is a transition from a vertex in A to a vertex in B . Due to reversibility of P , $Q(A, B) = Q(B, A)$. We will use the following inequalities, sometimes known as the *Expander Mixing Lemma for Inhomogeneous Graphs* (see e.g. [9, 16]). Let $A, B \subseteq V$, and $A^c = V \setminus A$, then

$$|Q(A, A^c) - \pi(A)\pi(A^c)| \leq \lambda\pi(A)\pi(A^c), \quad (3)$$

$$|Q(A, B) - \pi(A)\pi(B)| \leq \lambda\sqrt{\pi(A)\pi(B)\pi(A^c)\pi(B^c)}. \quad (4)$$

We also need lower bounds for Q^2 .

Lemma 1 *For any $A, B \subseteq V$, we have*

$$Q(A, B)^2 \geq (\pi(A)\pi(B))^2 - 2\lambda(\pi(A)\pi(B))^{3/2}(\pi(A^c)\pi(B^c))^{1/2}. \quad (5)$$

Proof.

$$\begin{aligned} Q(A, B)^2 &= ((Q(A, B) - \pi(A)\pi(B)) + \pi(A)\pi(B))^2 \\ &= (Q(A, B) - \pi(A)\pi(B))^2 + (\pi(A)\pi(B))^2 + 2\pi(A)\pi(B)(Q(A, B) - \pi(A)\pi(B)) \\ &\geq (\pi(A)\pi(B))^2 - 2\lambda\pi(A)\pi(B)\sqrt{\pi(A)\pi(B)\pi(A^c)\pi(B^c)}. \end{aligned} \quad (6)$$

The last line follows from (4). \square

Given $A, B \subseteq V$, define the quantity $R(A, B) = \sum_{x \in A} \pi(x)(P(x, B))^2$. This quantity is the expected change, in the stationary measure π , from A to B in one round of two-sample voting.

Lemma 2 For any $A \subseteq V$, we have $R(V, A) = Q_2(A, A)$, where Q_2 is the flow function for the two-step transition matrix P^2 .

Proof. From definition of $R(V, A)$, reversibility of P and $P^2(x, y) = \sum_{z \in V} P(x, z)P(z, y)$:

$$\begin{aligned} R(V, A) &= \sum_{x \in V} \pi(x) P(x, A)^2 = \sum_{x \in V} \pi(x) P(x, A) \sum_{y \in A} P(x, y) = \sum_{y \in A} \sum_{x \in V} \pi(x) P(x, A) P(x, y) \\ &= \sum_{y \in A} \sum_{x \in V} \pi(y) P(y, x) P(x, A) = \sum_{y \in A} \pi(y) \sum_{x \in V} P(y, x) P(x, A) \\ &= \sum_{y \in A} \pi(y) P^2(y, A) = Q_2(A, A). \end{aligned} \tag{7}$$

□

If G is a complete graph (with node loops), then $R(V, A) = \pi(A)^2 = (|A|/n)^2$ and $R(A, B) = \pi(A)\pi(B)^2 = |A| \cdot |B|^2/n^3$. The next two lemmas give bounds on deviations from these values in regular graphs.

Lemma 3 For $A \subseteq V$, we have

$$|R(V, A) - \pi(A)^2| = |Q_2(A, A^c) - \pi(A)\pi(A^c)| \leq \lambda^2 \pi(A)\pi(A^c). \tag{8}$$

Proof. By Lemma 2, $R(V, A) = Q_2(A, A)$, and standard manipulations give $Q_2(A, A) = Q_2(A, V) - Q_2(A, A^c) = \pi(A) - Q_2(A, A^c)$, so

$$R(V, A) - \pi(A)^2 = \pi(A) - Q_2(A, A^c) - \pi(A)^2 = \pi(A)\pi(A^c) - Q_2(A, A^c).$$

Taking the absolute value of both sides gives the first equality in (8). To obtain the inequality, apply (3) to P^2 , Q_2 and λ^2 as the second largest absolute eigenvalue of P^2 . □

Lemma 4 Let $A, B \subseteq V$, then

$$R(A, B) \geq \frac{Q(A, B)^2}{\pi(A)} \geq \pi(A)\pi(B)^2 - 2\lambda\pi(A)^{1/2}\pi(B)^{3/2}\pi(A^c)^{1/2}\pi(B^c)^{1/2}.$$

Proof. The second inequality is from Lemma 1. From convexity of the function $z \mapsto z^2$,

$$R(A, B) = \pi(A) \sum_{x \in A} \frac{\pi(x)}{\pi(A)} (P(x, B))^2 \geq \pi(A) \left(\sum_{x \in A} \frac{\pi(x)}{\pi(A)} P(x, B) \right)^2 = \frac{1}{\pi(A)} Q(A, B)^2. \tag{9}$$

□

Suppose the family of sets $\mathcal{C} = (A_1, \dots, A_k)$ is a partitioning of V . Define the quantity $S_{\mathcal{C}}(A) = \sum_{i=1}^k R(A, A_i)$. For a complete graph, $S_{\mathcal{C}}(A) = \sum_{i=1}^k \pi(A_i)^2$ and the following lemma bounds the deviation from this value in regular graphs.

Lemma 5 Consider a partitioning $\mathcal{C} = (A_1, \dots, A_k)$ of V . Then

$$\left| S_{\mathcal{C}}(V) - \sum_{i=1}^k \pi(A_i)^2 \right| \leq \lambda^2 \left(1 - \sum_{i=1}^k \pi(A_i)^2 \right).$$

Proof. Using Lemma 3, we get

$$\begin{aligned} \left| S_{\mathcal{C}}(V) - \sum_{i=1}^k \pi(A_i)^2 \right| &= \left| \sum_{i=1}^k R(V, A_i) - \pi(A_i)^2 \right| \leq \sum_{i=1}^k |R(V, A_i) - \pi(A_i)^2| \\ &\leq \sum_{i=1}^k \lambda^2 \pi(A_i) \pi(A_i^c) = \lambda^2 \left(1 - \sum_{i=1}^k \pi(A_i)^2 \right). \end{aligned}$$

□

Lemma 6 Let $\mathcal{C} = (A_1, \dots, A_k)$ be a partitioning of V . For any $A \subseteq V$,

$$\begin{aligned} S_{\mathcal{C}}(A) &\geq \pi(A) \sum_{i=1}^k \pi(A_i)^2 - 2\lambda \pi(A)^{1/2} \pi(A^c)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2}, \\ S_{\mathcal{C}}(A) &\leq \pi(A) \sum_{i=1}^k \pi(A_i)^2 + 2\lambda \pi(A)^{1/2} \pi(A^c)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2} + \lambda^2. \end{aligned} \tag{10}$$

Proof. Lemma 4 gives the first part:

$$S_{\mathcal{C}}(A) = \sum_{i=1}^k R(A, A_i) \geq \pi(A) \sum_{i=1}^k \pi(A_i)^2 - 2\lambda \pi(A)^{1/2} \pi(A^c)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2}.$$

For the second part, observe that $S_{\mathcal{C}}(A) + S_{\mathcal{C}}(A^c) = S_{\mathcal{C}}(V)$ and use Lemma 5 and (10):

$$\begin{aligned} S_{\mathcal{C}}(A) &= S_{\mathcal{C}}(V) - S_{\mathcal{C}}(A^c) \\ &\leq \sum_{i=1}^k \pi(A_i)^2 + \lambda^2 \left(1 - \sum_{i=1}^k \pi(A_i)^2 \right) - \pi(A^c) \sum_{i=1}^k \pi(A_i)^2 + 2\lambda \pi(A^c)^{1/2} \pi(A)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2} \\ &= \pi(A) \sum_{i=1}^k \pi(A_i)^2 + 2\lambda \pi(A)^{1/2} \pi(A^c)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2} + \lambda^2 \left(1 - \sum_{i=1}^k \pi(A_i)^2 \right). \end{aligned}$$

□

3 Proof of Theorem 1

From now on we assume the graph is d -regular, so $\pi(x) = 1/n$, and for $A \subseteq V$, $\pi(A) = |A|/n$. Furthermore, $nR(A, B) = \sum_{x \in A} (d_B(x)/d)^2$ is the expected number of vertices in A which pick two opinions in B . When clear from the context, we use A instead of $|A|$ for the size of A .

Let A_j be the set of vertices with opinion j . At any step, the opinions are ordered according to their sizes: $A_1 \geq A_2 \geq \dots \geq A_k$. Thus $\mathcal{C} = \{A_1, \dots, A_k\}$ is a partition of V . Let A'_j be the set of vertices

with opinion j after one round. We have the following equality, where the second term in (11) is the expected number of vertices changing their opinion to A_j and the third term is the expected number of vertices changing their opinion from A_j .

$$\mathbf{E}(\pi(A'_j)|\mathcal{C}) = \pi(A_j) + R(V \setminus A_j, A_j) - \sum_{i \neq j} R(A_j, A_i) \quad (11)$$

$$\begin{aligned} &= \pi(A_j) + R(V, A_j) - R(A_j, A_j) - \sum_{i \neq j} R(A_j, A_i) \\ &= \pi(A_j) + R(V, A_j) - S_{\mathcal{C}}(A_j). \end{aligned} \quad (12)$$

The next lemma shows that, given a sufficient advantage of opinion 1, after one round of voting opinion 1 remains the largest opinion. More precisely, the lemma gives lower bounds on the increase of the size of opinion 1 and on the increase of the advantage of this opinion over the other opinions.

Lemma 7 *Assume $A_1 \leq 2n/3$, $A_1 - A_2 \geq Cn\sqrt{(\log n)/A_1}$ (requiring $A_1 \geq C^{2/3}n^{2/3}\log^{1/3} n$), where $C = 240\sqrt{2}$, and $\lambda \leq (A_1 - A_2)/(32n)$. Then with probability at least $1 - 1/n^2$,*

$$A'_1 \geq A_1 \left(1 + \frac{A_1 - A_2}{5n}\right). \quad (13)$$

$$\min_{2 \leq j \leq k} \{A'_1 - A'_j\} \geq (A_1 - A_2) \left(1 + \frac{A_1}{10n}\right), \quad (14)$$

Proof. Several times in this proof we use that $\pi(A_1) \leq 2/3$, which implies that $\pi(A_1^c) \geq 1/3$. Our proof uses concepts from Bechetti et. al. [3, 4] and makes extensive use of the following Chernoff bounds. If X is the sum of independent Bernoulli random variables, then for $\varepsilon \in (0, 1)$ and $\delta \geq 1$,

$$\Pr(X \geq (1 + \varepsilon)\mathbf{E}(X)), \Pr(X \leq (1 - \varepsilon)\mathbf{E}(X)) \leq \exp(-\varepsilon^2\mathbf{E}(X)/3), \quad (15)$$

$$\Pr(X \geq (1 + \delta)\mathbf{E}(X)) \leq \exp(-\delta\mathbf{E}(X)/3). \quad (16)$$

From Equation (12) and Lemmas 3 and 6, we have the following lower and upper bounds on $\mathbf{E}(\pi(A'_j)|\mathcal{C})$ for any $j \in [k]$.

$$\begin{aligned} \mathbf{E}(\pi(A'_j)|\mathcal{C}) &= \pi(A_j) + R(V, A_j) - S_{\mathcal{C}}(A_j) \\ &\geq \pi(A_j) + \pi(A_j)^2 - \lambda^2\pi(A_j)\pi(A_j^c) \\ &\quad - \pi(A_j) \sum_{i=1}^k \pi(A_i)^2 - 2\lambda\pi(A_j)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2} - \lambda^2 \\ &\geq \pi(A_j) \left(1 + \pi(A_j) - \sum_{i=1}^k \pi(A_i)^2\right) - 2\lambda\pi(A_j)^{1/2}\pi(A_1)^{1/2} - (5/4)\lambda^2. \end{aligned} \quad (17)$$

$$\begin{aligned} \mathbf{E}(\pi(A'_j)|\mathcal{C}) &= \pi(A_j) + R(V, A_j) - S_{\mathcal{C}}(A_j) \\ &\leq \pi(A_j) + \pi(A_j)^2 + \lambda^2\pi(A_j)\pi(A_j^c) - \pi(A_j) \sum_{i=1}^k \pi(A_i)^2 + 2\lambda\pi(A_j)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2} \\ &\leq \pi(A_j) \left(1 + \pi(A_j) - \sum_{i=1}^k \pi(A_i)^2\right) + (1/4)\lambda^2 + 2\lambda\pi(A_j)^{1/2}\pi(A_1)^{1/2}. \end{aligned} \quad (18)$$

By assumption, $\lambda \leq \pi(A_1)/32$ and $\pi(A_1) \leq 2/3$, so (17) and (18) imply

$$\pi(A_1)/2 \leq \mathbf{E}(\pi(A'_1)|\mathcal{C}) \leq 2\pi(A_1). \quad (19)$$

Define $\varepsilon_1 = \sqrt{\frac{9 \log n}{\mathbf{E}(A'_1|\mathcal{C})}} \leq \sqrt{\frac{18 \log n}{A_1}} < 1$. Therefore, using the Chernoff bounds (15), we get

$$\Pr(A'_1 \leq (1 - \varepsilon_1)\mathbf{E}(A'_1|\mathcal{C})|\mathcal{C}) \leq e^{-3 \log(n)} = n^{-3}. \quad (20)$$

For a fixed j , $2 \leq j \leq k$, define $\varepsilon_j = \sqrt{9(\log n)\mathbf{E}(A'_1|\mathcal{C})/\mathbf{E}(A'_j|\mathcal{C})}$. We show that

$$\Pr(A'_j \geq (1 + \varepsilon_j)\mathbf{E}(A'_j|\mathcal{C})|\mathcal{C}) \leq n^{-3}. \quad (21)$$

Indeed, if $\varepsilon_j \leq 1$, then the Chernoff bounds (15) give

$$\Pr(A'_j \geq (1 + \varepsilon_j)\mathbf{E}(A'_j|\mathcal{C})|\mathcal{C}) \leq e^{-3(\log n)\mathbf{E}(A'_1|\mathcal{C})/\mathbf{E}(A'_j|\mathcal{C})} \leq e^{-3 \log(n)} = n^{-3}.$$

If $\varepsilon_j > 1$, then the Chernoff bound (16) gives

$$\Pr(A'_j \geq (1 + \varepsilon_j)\mathbf{E}(A'_j|\mathcal{C})|\mathcal{C}) \leq e^{-\sqrt{(\log n)\mathbf{E}(A'_1|\mathcal{C})}} \leq e^{-\sqrt{A_1}} \leq e^{-(Cn)^{1/3}} \leq n^{-3}.$$

The bounds (20) and (21) imply that with probability at least $1 - kn^{-3} \geq 1 - n^{-2}$, for all $2 \leq j \leq k$,

$$\begin{aligned} A'_1 - A'_j &\geq (1 - \varepsilon_1)\mathbf{E}(A'_1|\mathcal{C}) - (1 + \varepsilon_j)\mathbf{E}(A'_j|\mathcal{C}) \\ &= \mathbf{E}(A'_1 - A'_j|\mathcal{C}) - 2\sqrt{9(\log n)\mathbf{E}(A'_1|\mathcal{C})} \end{aligned} \quad (22)$$

and thus

$$\pi(A'_1) - \pi(A'_j) \geq \mathbf{E}(\pi(A'_1) - \pi(A'_j)|\mathcal{C}) - 2\sqrt{\frac{9(\log n)\mathbf{E}(\pi(A'_1)|\mathcal{C})}{n}}. \quad (23)$$

The right-hand side of (18) is non-increasing with increasing j , so for each $2 \leq j \leq k$,

$$\mathbf{E}(\pi(A'_j)|\mathcal{C}) \leq \pi(A_2)(1 + \pi(A_2) - \sum_{i=1}^k \pi(A_i)^2) + \lambda^2 + 2\lambda\pi(A_1). \quad (24)$$

Let $\Delta = \pi(A_1) - \pi(A_2)$. Inequalities (17) and (24) give for each $2 \leq j \leq k$,

$$\begin{aligned} \mathbf{E}(\pi(A'_1) - \pi(A'_j)|\mathcal{C}) &\geq \pi(A_1) \left(1 + \pi(A_1) - \sum_{i=1}^k \pi(A_i)^2 \right) - 2\lambda\pi(A_1) - (5/4)\lambda^2 \\ &\quad - \left(\pi(A_2) \left(1 + \pi(A_2) - \sum_{i=1}^k \pi(A_i)^2 \right) + (1/4)\lambda^2 + 2\lambda\pi(A_1) \right) \\ &= \Delta \left(1 + \pi(A_1) + \pi(A_2) - \sum_{i=1}^k \pi(A_i)^2 \right) - 4\lambda\pi(A_1) - (3/2)\lambda^2 \\ &\geq \Delta(1 + \pi(A_1)\pi(A_1^c)) - 4\lambda\pi(A_1) - 2\lambda^2 \\ &\geq \Delta + \Delta\pi(A_1)/7. \end{aligned} \quad (25)$$

$$\geq \Delta + \Delta\pi(A_1)/7. \quad (26)$$

Inequality (25) holds because $\sum_{i=2}^k \pi(A_i)^2 \leq \pi(A_2)$. In the last step we used that $\pi(A_1^c) \geq 1/3$ and $\lambda \leq \Delta/32$. From (23), (26) and (19), with probability at least $1 - n^{-2}$,

$$\begin{aligned} \min_{2 \leq j \leq k} \{\pi(A'_1) - \pi(A'_j)\} &\geq \mathbf{E}(\pi(A'_1) - \pi(A'_j)|\mathcal{C}) - \frac{\varepsilon_1}{n} \mathbf{E}(A'_1|\mathcal{C}) - \frac{\varepsilon_j}{n} \mathbf{E}(A'_j|\mathcal{C}) \\ &\geq \Delta(1 + \pi(A_1)/7) - 6\sqrt{\frac{2 \log n}{n} \pi(A_1)} \\ &= \Delta \left(1 + \pi(A_1)/7 - \frac{6}{\Delta} \sqrt{\frac{2 \log n}{n} \pi(A_1)} \right). \end{aligned}$$

By assumption, $\Delta \geq 240\sqrt{2 \log(n)/A_1}$, so with probability at least $1 - n^{-2}$,

$$\min_{2 \leq j \leq k} \{\pi(A'_1) - \pi(A'_j)\} \geq \Delta(1 + \pi(A_1)/10), \quad (27)$$

and we get we get (14). This also proves that w.h.p. opinion 1 remains the majority opinion. The order between the other opinions might change.

To get information about the increase in the number of vertices with opinion 1, we use Equation (17) with $j = 1$ and the assumption that $\lambda \leq \Delta/32$. We obtain

$$\begin{aligned} \mathbf{E}(\pi(A'_1)|\mathcal{C}) &\geq \pi(A_1)(1 + \pi(A_1) - \sum_{i=1}^k \pi(A_i)^2) - \Delta\pi(A_1)/16 - \Delta^2/(32)^2 \\ &\geq \pi(A_1)(1 + \pi(A_1) - \pi(A_1)^2 - \pi(A_2)\pi(A_1^c) - \Delta/16 - \Delta/(32)^2) \\ &> \pi(A_1)(1 + \Delta/4). \end{aligned} \quad (28)$$

By using Chernoff bounds (15) with $\varepsilon = \sqrt{\frac{9 \log n}{\mathbf{E}(A'_1|\mathcal{C})}}$ and Inequalities (28) and (19), with probability at least $1 - n^{-2}$,

$$\begin{aligned} A'_1 &\geq A_1(1 + \Delta/4) - \sqrt{\mathbf{E}(A'_1|\mathcal{C})9 \log n} \geq A_1(1 + \Delta/4) - \sqrt{18A_1 \log n} \\ &= A_1(1 + \Delta/4 - 3\sqrt{2}\sqrt{\log n/A_1}). \end{aligned} \quad (29)$$

From the assumptions of the lemma, we have $\Delta/20 = (A_1 - A_2)/(20n) \geq 3\sqrt{2}\sqrt{\log n/A_1}$. Therefore (29) implies $A'_1 \geq A_1(1 + \Delta/5)$, which is the same as (13). \square

Lemma 8 *Assume $A_1 \leq 2n/3$, $A_1 - A_2 \geq Cn\sqrt{(\log n)/A_1}$ (requiring $A_1 \geq C^{2/3}n^{2/3}\log^{1/3}n$), where $C = 240\sqrt{2}$, and $\lambda \leq (A_1 - A_2)/(32n)$. Then with probability at least $1 - 1/n$, after at most $O((n/A_1)\log(A_1/(A_1 - A_2)))$ rounds, the number of vertices with opinion 1 is at least $2n/3$.*

Proof. We apply Lemma 7 to consecutive rounds until the size of opinion 1 reaches $2n/3$. Since w.h.p. the difference between the size of opinion 1 and the size of the second largest opinion increases, our assumption about λ in Lemma 7 is maintained from round to round. At the end of each round the ordering of the opinions according to their sizes can change. In that case we exchange the labels of the opinions so that $A_1(t) \geq A_2(t) \cdots \geq A_k(t)$ for every round t . Lemma 7, however, implies that w.h.p. opinion 1 remains the largest opinion, so it is not relabeled.

Denote by $x(i)$ the fraction of vertices with opinion 1 at the end of round i , where $x(0) = \pi(A_1)$, and by $y(i)$ the difference between the fraction of vertices with opinion 1 and the fraction of vertices with

the second largest opinion, where $y(0) = \Delta = \pi(A_1) - \pi(A_2) < x(0)$. By (13) and (14) and induction on the number of rounds, with probability at least $1 - 1/n$, for each round $1 \leq i \leq n$, if $x(i) < 2/3$, then

$$x(i) \geq x(i-1)(1 + y(i-1)/5), \quad (30)$$

$$y(i) \geq y(i-1)(1 + x(i-1)/10). \quad (31)$$

Iterating (30) and (31) for $j = \lceil 10/x(0) \rceil < n$ rounds, we get $y(j) \geq 2y(0)$ and $x(j) \geq x(0) + y(0)$, or $x(i) \geq 2/3$ for some $i \leq j$. Repeating this $r = \lceil \log_2(x(0)/y(0)) \rceil$ times, we get for round $i_1 = rj < n$, $y(i_1) \geq x(0)$ and $x(i_1) \geq x(0) + y(0) + 2y(0) + 4y(0) \cdots + 2^{r-1}y(0) \geq 2x(0)$, or $x(i) \geq 2/3$ for some $i \leq i_1$.

If for some $q \geq 1$, $y(i_q) \geq 2^{q-1}x(0)$ and $x(i_q) \geq 2^q x(0)$, or $x(i) \geq 2/3$ for some $i \leq i_q$, then at the end of round $i_{q+1} = i_q + \lceil 10/(2^q x(0)) \rceil$, $y(i_{q+1}) \geq 2^q x(0)$ and $x(i_{q+1}) \geq 2^{q+1} x(0)$, or $x(i) \geq 2/3$ for some $i \leq i_{q+1}$, or $i_{q+1} > n$. Taking $q = \lceil \log_2(1/x(0)) \rceil$, we have $i_q = O((1/x(0)) \log(x(0)/y(0))) = O((n/A_1) \log(A_1/(A_1 - A_2)))$ (observe that $i_q < n$) and $2^q x(0) \geq 1$, so we must have $x(i) \geq 2/3$ for some $i \leq i_q$. \square

When the largest opinion reaches the size $2n/3$, it will take over the whole graph within additional $O(\log n)$ rounds. The progress of voting in this final stage would be slowest, if all minority opinions were joined together into a single “second” opinion. The proof of the next lemma follows the proof from [9] that two-sample voting finishes in $O(\log n)$ rounds, if there are two opinions, the majority opinion has size at least cn , for a constant $c > 1/2$, and λ is sufficiently small.

Lemma 9 *Let G be a connected regular graph with $\lambda \leq 1/4$. If the majority opinion has size at least $2n/3$, then with probability at least $1 - n^{-2}$, the voting finishes within $\mathcal{O}(\log n)$ rounds.*

Proof. Let A represent the current set of vertices with the majority opinion. We put all minority opinions into one opinion set $B = V \setminus A$ and analyse two-sample voting with these two opinions. The majority opinion in this process is always a subset of the majority opinion in the original process, when there are distinct minority opinions.

Let A' and B' be the corresponding sets in the next round. We compute $\mathbf{E}(A'|A)$. Observe that since in our context $\mathcal{C} = (A, B)$ and $S_{\mathcal{C}}(A) = R(A, A) + R(B, A)$, then, from Equation (12), we have

$$\begin{aligned} \mathbf{E}(\pi(B')|B) &= \pi(B) + R(V, B) - S_{\mathcal{C}}(B) \\ &\leq \pi(B) + \pi(B)^2 + \lambda^2 \pi(B) \pi(A) - \sum_{x \in B} \pi(x) (P(x, A)^2 + P(x, B)^2) \\ &\leq \pi(B) + \pi(B)^2 + \lambda^2 \pi(B) \pi(A) - \pi(B)/2 \\ &= \pi(B) + \pi(B)(1/2 - (1 - \lambda^2)\pi(A)). \end{aligned} \quad (32)$$

Given $\lambda \leq 1/4$ and $\pi(A) \geq 2/3$, (32) implies

$$\mathbf{E}(\pi(B')|B) \leq (7/8)\pi(B). \quad (33)$$

A standard coupling shows that if $B_1 \subseteq B_2$, then $\mathbf{Pr}(\pi(B') \geq \delta | B = B_1) \leq \mathbf{Pr}(\pi(B') \geq \delta | B = B_2)$. Take arbitrary sets $B_1 \subseteq B_2 \subseteq V$ such that $\pi(B_1) \leq 1/3$ and $\pi(B_2) = 1/3$, and apply Hoeffding's

Inequality to get

$$\begin{aligned}
\Pr((\pi(B') \geq 1/3) \mid B = B_1) &\leq \Pr(\pi(B') \geq 1/3 \mid B = B_2) \\
&= \Pr(|B'| \geq n/3 \mid B = B_2) \\
&\leq \Pr(|B'| \geq \mathbf{E}(|B'| \mid B = B_2) + n/24 \mid B = B_2) \\
&\leq e^{-2(n/24)^2/n} = o(n^{-10}).
\end{aligned} \tag{34}$$

Inequality (34) holds because $\mathbf{E}(|B'| \mid B = B_2) \leq (7/8)\pi(B_2) = (7/24)n$, and (35) follows from Hoeffding's Inequality. The bound above implies that in the next n rounds, the probability to have a minority with more than $n/3$ opinions is $o(n^{-9})$.

Let B_t be the set with the minority opinion after t rounds of this final stage of voting. We assume that B_0 is a fixed set such that $|B_0| \leq (1/3)n$. To obtain the claimed bound on the number of rounds, we use (33) and (35) in a straightforward application of Markov's Inequality:

$$\Pr(B_t \neq \emptyset) = \Pr(\pi(B_t) \geq 1/n) \leq n \cdot \mathbf{E}(\pi(B_t)). \tag{36}$$

Using (33), for each $t \geq 1$,

$$\mathbf{E}(\pi(B_t) \mid B_{t-1}) \leq \begin{cases} (7/8)\pi(B_{t-1}), & \text{if } B_{t-1} \leq 1/3, \\ 1, & \text{if } B_{t-1} > 1/3. \end{cases}$$

This gives

$$\mathbf{E}(\pi(B_t)) = \mathbf{E}(\mathbf{E}(\pi(B_t) \mid B_{t-1})) \leq (7/8)\mathbf{E}(\pi(B_t)) + \Pr(B_{t-1} > 1/3),$$

Applied the above inequality iteratively to obtain

$$\mathbf{E}(\pi(B_t)) \leq (7/8)^t \pi(B_0) + \sum_{\tau=0}^{t-1} \Pr(B_\tau > 1/3) \leq (1/3) \cdot (7/8)^t + o(n^{-8}).$$

Thus for $T = K \log n$ with $K = 4/\log(8/7)$, $\mathbf{E}(\pi(B_T)) \leq n^{-3}$, so (36) implies that with probability at least $1 - n^{-2}$, B_T is empty, that is, the voting finishes in $K \log(n)$ rounds. \square

4 Reducing Three-sample voting to Two-sample voting

In this section we study the three-sample voting process, which is similar to the two-sample voting process but samples three neighbours in each round. Additionally, if all three opinions are distinct, the vertex adopts the opinion of the first sampled neighbour. Formally, each vertex v selects three random neighbours with replacement and considers their opinions, say, $Y_{v,1}, Y_{v,2}, Y_{v,3}$. Vertex v changes its opinion to the majority of $\{Y_{v,1}, Y_{v,2}, Y_{v,3}\}$, or, if there is no majority, to $Y_{v,1}$. Suppose in a given round we have k opinions. Let $\mathcal{C} = (A_1, \dots, A_k)$ be the partition of the vertices given by the opinions, where A_j is the set of vertices with opinion j . Let A_j'' be the vertices with opinion j at the next round. Moreover, let A_j' be the set of vertices v such that $Y_{v,1} = j$.

The following lemma will allow us to use the results of Lemma 8 and Lemma 9 for the three-sample protocol. Due to space restrictions the proof of the lemma, and the explanation of its application in Lemma 8 and Lemma 9 is given in the Appendix.

Lemma 10 *Let G be a connected graph and let $\mathcal{C} = (A_1, \dots, A_k)$ partition V . Then*

$$\mathbf{E}(\pi(A_j'')|\mathcal{C}) = \pi(A_j) + R(V, A_j) - \mathbf{E}(S_{\mathcal{C}}(A_j')|\mathcal{C}) \quad (37)$$

Moreover,

$$\mathbf{E}(S_{\mathcal{C}}(A_j')|\mathcal{C}) \geq \pi(A_j) \sum_{i=1}^k \pi(A_i)^2 - 2\lambda\pi(A_j)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2} \quad (38)$$

and

$$\mathbf{E}(S_{\mathcal{C}}(A_j'^c)|\mathcal{C}) \geq \pi(A_j^c) \sum_{i=1}^k \pi(A_i)^2 - 2\lambda\pi(A_j^c)^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2}. \quad (39)$$

If $\mathcal{C} = (A, B)$, then

$$\mathbf{E}(S_{\mathcal{C}}(B')|\mathcal{C}) \geq \pi(B)/4. \quad (40)$$

Before proving Lemma 10, we observe that Equations (38) and (39) are enough to get Lemma 6 for the values $\mathbf{E}(S_{\mathcal{C}}(A_j')|\mathcal{C})$, i.e. the bounds we got for $S_{\mathcal{C}}(A_j)$ are also valid for $\mathbf{E}(S_{\mathcal{C}}(A_j')|\mathcal{C})$. Our proof of Lemma 8 is based on the concentration of sums of Bernoulli random variables around their expected values, but we see that the expected values, or, more precisely, the respective bounds on those values, are the same in both protocols. Thus the “w.h.p.” result of Lemma 8 applies also to the three-sample voting model. The same argument but using Equation (40) allows us to transfer the result of Lemma 9 from the two-sample to the three-sample voting.

Proof. First of all, observe that A_j' is the result of choosing only one vertex, i.e. one round of standard pull voting. For given vertex v this accounts for $Y_{v,1}$. We now consider $Y_{v,2}, Y_{v,3}$ taken in the original partition \mathcal{C} . Observe that given A_j' , then A_j'' is the set of vertices in A_j' such that the other two opinions taken in the original partition \mathcal{C} are not equal to any opinion i other than j , plus the set of vertices outside A_j' such that the other two opinions in \mathcal{C} are equal to j . Therefore

$$\pi(A_j'') = \pi(A_j') + \pi(\{x \in A_j'^c : Y_{x,2} = Y_{x,3} = j\}) - \pi(\{x \in A_j' : Y_{x,2} = Y_{x,3} = i, i \neq j\}) \quad (41)$$

By a result of [14] for classical pull voting, we have $\pi(A_j'|\mathcal{C}) = \pi(A_j)$. From there, it is relatively straightforward to get that

$$\begin{aligned} \mathbf{E}(\pi(A_j'')|\mathcal{C}) &= \pi(A_j) + \mathbf{E} \left(\sum_{x \in A_j'^c} \pi(x) P(x, A_j)^2 - \sum_{x \in A_j'} \pi(x) \sum_{i \neq j} P(x, A_i)^2 \middle| \mathcal{C} \right) \\ &= \pi(A_j) + \mathbf{E} \left(\sum_{x \in V} \pi(x) P(x, A_j)^2 - \sum_{x \in A_j'} \pi(x) \sum_{i=1}^k P(x, A_i)^2 \middle| \mathcal{C} \right) \\ &= \pi(A_j) + R(V, A_j) - \mathbf{E}(S_{\mathcal{C}}(A_j')|\mathcal{C}) \end{aligned} \quad (42)$$

For the lower bound in (38) we use Lemma 6 to get

$$S_{\mathcal{C}}(A_j') \geq \pi(A_j') \sum_{i=1}^k \pi(A_i)^2 - 2\lambda\pi(A_j')^{1/2} \sum_{i=1}^k \pi(A_i)^{3/2}. \quad (43)$$

By concavity of $f(x) = x^{1/2}$ we have

$$\mathbf{E}(\pi(A_j')^{1/2}|\mathcal{C}) \leq (\mathbf{E}(\pi(A_j')|\mathcal{C}))^{1/2} = (\pi(A_j))^{1/2},$$

obtaining the result of Equation (38). A similar argument gives us the result of Equation (39). \square

References

- [1] D. Aldous. Meeting times for independent Markov chains. *Stochastic Processes and their Applications* 38(2):185–193, (1991).
- [2] D. Aldous and J. Fill. *Reversible Markov Chains and Random Walks on Graphs*, <http://stat-www.berkeley.edu/pub/users/aldous/RWG/book.html>.
- [3] L. Becchetti, A. Clementi, E. Natale, F. Pasquale, R. Silvestri and L. Trevisan. Simple dynamics for plurality consensus. In *Proceedings of the 26th ACM symposium on Parallelism in Algorithms and Architectures (SPAA '14)*, pages 247–256, (2014).
- [4] L. Becchetti, A. Clementi, E. Natale, F. Pasquale, R. Silvestri and L. Trevisan. Simple dynamics for plurality consensus. [arXiv:1310.2858v3](https://arxiv.org/abs/1310.2858v3)
- [5] S. Brahma, S. Macharla, S. P. Pal, S. R. Singh. Fair Leader Election by Randomized Voting. In *ICDCIT 2004*, pages 22–31, (2004).
- [6] A. Broder, A. Frieze, S. Suen and E. Upfal. Optimal construction of edge disjoint paths in random graphs. *SIAM Journal on Computing*, 28(2), pages 541–573, (1999).
- [7] C. Cooper, R. Elsässer, H. Ono, T. Radzik. Coalescing Random Walks and Voting on Connected Graphs. *SIAM Journal of Discrete Math*, 27, pages 1748–1758, (2013).
- [8] C. Cooper, R. Elsässer and T. Radzik. The power of two choices in distributed voting, *ICALP 2014*, pages 435–446, (2014).
- [9] C. Cooper, R. Elsässer, T. Radzik, N. Rivera and T. Shiraga. Fast consensus for voting on general expander graphs. In *DISC 2015 – 29th International Symposium on Distributed Computing*, Springer-Verlag LNCS 9363, pages 248–262. (2015).
- [10] J. T. Cox. Coalescing random walks and voter model consensus times on the torus in \mathbb{Z}^d . *The Annals of Probability* 17(4):1333–1366, (1989).
- [11] X. Deng and C. Papadimitriou. On the Complexity of Cooperative Solution Concepts. *Mathematics of Operations Research* 19, pages 257–266, (1994).
- [12] J. Friedman. A proof of Alon’s second eigenvalue conjecture. In *STOC 2003: Proc. 35th Annual ACM Symposium on Theory of Computing*, pages 720–724, (2003).
- [13] D. Gifford. Weighted Voting for Replicated Data. In *SOSP 1979: Proceedings of the 7th ACM Symposium on Operating Systems Principles*, pages 150–162, (1979).
- [14] Y. Hassin and D. Peleg. Distributed probabilistic polling and applications to proportionate agreement. *Information & Computation*, 171, pages 248–268, (2001).
- [15] B. Johnson, *Design and Analysis of Fault Tolerant Digital Systems*, Addison-Wesley, (1989).
- [16] L. Trevisan. The Expander Mixing Lemma in Irregular Graphs. <https://lucatrevisan.wordpress.com/2014/08/26>